SHARP

2022 Wits Mathematics Competition
Qualifying Round
Junior Secondary

Instructions

This exam consists of 20 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The mark allocation is as follows:

Questions 1-5 are each worth 3 points,
Questions 6-10 are each worth 4 points,
Questions 11-15 are each worth 5 points,
Questions 16-20 are each worth 6 points.
The total number of points available is 90.

The time limit on this exam is 75 minutes, calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed. If you are using the computer-friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profoundity, he is talking nonsense" — Alfred North Whitehead

A. 3 point questions

1 A perfect number is an integer that is equal to the sum of all of its positive divisors, except itself. For example, 28 is a perfect number because $28 = 1 + 2 + 4 + 7 + 14$. Which of the following is a perfect number?

(A) 10 (B) 13 (C) 6 (D) 8 (E) 9

Solution Option: C It is well known that 6 and 28 are the first two perfect numbers. However, for those who don't know this $6 = 1+2+3$. None of the other options work.

2 The smallest number in the set $\{0, -17, 4, 3, -2\}$ is

(A) 4 (B) -17 (C) -2 (D) 0 (E) 3

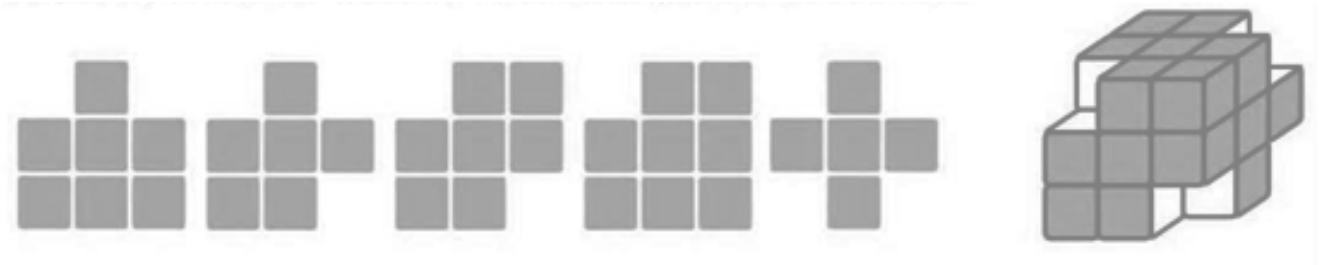
Solution Option: B. $-17 < -2 < 0 < 3 < 4$.

3 What is the minimum number of digits that must be removed from the number 12323314, so that the resulting number is the same when read from either left to right or right to left?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution Option: C First notice that the 4 has to be removed (or else everythign else removed with is inefficent). This leaves us with 1232331 and it's easy to check that removing a single extra digit here isn't enough and that two is enough (the last two threes or both twos for example).

4 Tumelo makes a large cube from 27 small white cubes. She paints all the faces of the large cube (shaded in the picture below). She then removes a small cube from four corners, as shown. While the paint is still wet, she stamps each of the new faces onto a piece of paper. How many of the following stamps can Tumelo make?



(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution Option: D. All but the last one can be made. To see that the last one can't be made observe that no side has only 5 remaining squares.

5 The weight limit for an elevator is 1500 kilograms. The average weight of the people in the elevator is 80 kilograms. If the combined weight of the people is 100 kilograms over the limit, how many people are in the elevator?

(A) 14 (B) 17 (C) 16 (D) 20 (E) 13

Solution Option: D The weight of the people in the elevator is 1600kg and since the average weight is 80kg, there must be 20 people.

B. 4 point questions

6 Callan has 10 blue marbles, 6 green marbles and 5 red marbles in a bag. He draws one marble at a time and puts it aside. How many times should he draw to ensure that he has at least one marble of each colour?

(A) 4 (B) 12 (C) 15 (D) 16 (E) 17

Solution Option: E Worst case scenario is that Callan first draws all 10 blue and 6 green marbles and hence still doesn't have one of each colour. Drawing a 17th marble forces one of each colour.

7 Compute $1000 - 999 + 998 - 997 + \cdots + 4 - 3 + 2 - 1$.

(A) 999 (B) 500 (C) 250 (D) 2 (E) 1

Solution Option: B Note that if you pair the numbers from the start, the sum in each pair, or rather the difference in each pair, is 1. Hence, the sum is 500 as there are 500 pairs which has a difference of 1.

8 Find x where $2^x - 2^{(x-3)} = 896$.

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution Option: D $2^x - 2^{(x-3)} = 2^{(x-3)}(2^3 - 1) = 2^{(x-3)} \times 7 = 896$.
Hence $2^{(x-3)} = 128 = 2^7$ and thus $x = 10$.

9 On each of the sides of triangle ABC is constructed a semi-circle on the outside of the triangle with diameters AB, BC and CA respectively. If the three semi-circles have areas 8π, $\frac{25\pi}{2}$ and 18π, what is the perimeter of triangle ABC?

(A) 18 (B) 20 (C) 24 (D) 30 (E) $\frac{77}{2}$

Solution Option: D If the semi-circles were completed to be full circles, the circles have area 16π, 25π and 36π. Therefore the semi-circles have radii 4, 5 and 6 respectively and hence the perimeter of the triangle is therefore $2 \times (4 + 5 + 6) = 30$.

10 Suppose that x^* means $\frac{1}{x}$, the reciprocal of x. For example, $5^* = \frac{1}{5}$. How many of the following statements are true?

(i) $2^* + 4^* = 6^*$ (ii) $3^* \times 5^* = 15^*$ (iii) $7^* - 3^* = 4^*$ (iv) $12^* \div 3^* = 4^*$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution Option: C

(i) $\frac{1}{2} + \frac{1}{4} = \frac{3}{4} \neq \frac{1}{6}$
(ii) $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$
(iii) $\frac{1}{7} - \frac{1}{3} = \frac{-4}{21} \neq \frac{1}{4}$
(iv) $\frac{1}{12} \div \frac{1}{3} = \frac{1}{4}$

C. 5 point questions

11 When the product $5^3 \times 7^{52}$ is expanded, the units digit is

(A) 5 (B) 3 (C) 9 (D) 7 (E) 0

Solution Option: A Since the answer is odd and is a multiple of 5, the units digit will be 5.

12 A number is *Beprisque* if it is the only natural number (positive whole number) between a prime number and a perfect square (e.g. 8 and 10 are Beprisque but 12 is not). The number of two-digit Beprisque numbers (including 10) is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution Option: E Since there are less square numbers than primes below 100, we just test each square number. The perfect squares are 9, 16, 25, 36, 49, 64, 81, 100 and we need to find how many of these are such that the number 2 away from them is a prime number. None of the even perfect squares can work as the number 2 away from them cannot be prime. Hence we are left with 9, 25, 49, 81. The numbers 11, 23, 47, 79 and 83 are all prime and form a Beprisque number. Thus 5.

13 How many 3 digit numbers (whole numbers between 100 and 999) are multiples of 6, 10 **and** 15?

(A) 18 (B) 20 (C) 30 (D) 33 (E) 60

Solution Option: C A number is a multiple of all three of these if and only if it's a multiple of 30 (the lcm of the three numbers). Which means the set we're looking at is 120, 150, 180, ...,960,990. 990 is the 33^{rd} multiple of 30 but we must then take away 30, 60 and 90 leaving 30 elements in this set.

14 $ABCD$ is a rectangle. Point E lies on AB such that angle $DEC = 90^o$. $DC = \sqrt{10}$ cm and $DE = 3$cm. Find the area of $ABCD$.

(A) $2\sqrt{10}$ (B) $\frac{3\sqrt{3}}{2}$ (C) 3 (D) 6 (E) 12

Solution Option: C Solution 1:
From Pythagoras, $EC = \sqrt{10-9} = 1$ and hence area of triangle DEC is $\frac{3}{2}$. Hence area of the rectangle $ABCD$ is double and is 3.

Solution 2:
Observe that ADE, DEC and BEC are similar triangles to get $BC = \frac{3}{\sqrt{10}}$.

15 There are seven points on a piece of paper. Exactly four of these points are on a straight line. No other line contains more than two of these points. Three of these seven points are selected to form the vertices of a triangle. How many triangles are possible?

(A) 18 (B) 28 (C) 30 (D) 31 (E) 33

Solution Option: D We have three options for our triangles. Two of the vertices are from the 4 points on a straight line, or only one of them is or none of them are. When none of them are, there is only one option. When one of them is there are 4×3 options. When two of them are there are 6×3 options. Total is $1 + 12 + 18 = 31$.

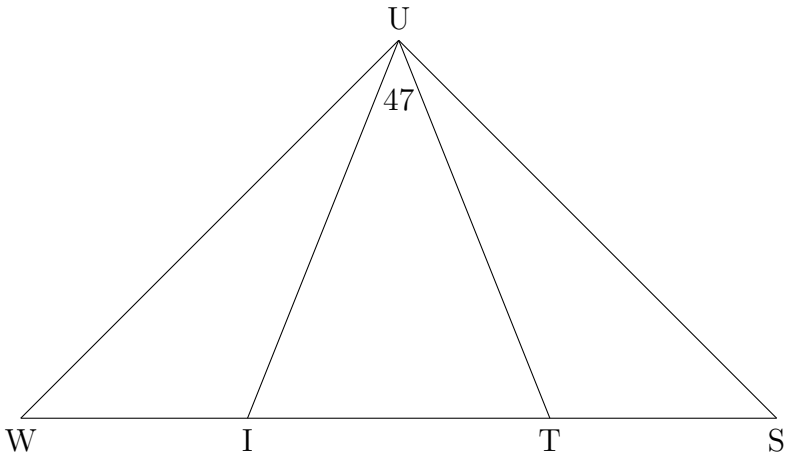
D. 6 point questions

16 How many zeroes will there be at the end of $1 \times 2 \times 3 \times 4 \times \cdots \times 39 \times 40$ when multiplied out (when all the natural numbers from 1 to 40 are multiplied together)?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 11

Solution Option: D Each multiple of 5 adds an extra zero to the end and each multiple of 25 adds an additional zeroe. Hence $8 + 1 = 9$.

17 In a triangle UWS the points I and T are placed on side WS so that $WU = WT$ and $SU = SI$. Determine $W\hat{U}S$ if $I\hat{U}T = 47^o$



(A) 82 (B) 84 (C) 86 (D) 88 (E) 90

Solution Option: C Set $W\hat{T}U = x$. Then angle chase to get, $W\hat{U}I = x - 47$, $T\hat{I}U = 133 - x$ and $T\hat{U}S = 86 - x$. Adding the three angles around U together gives a final answer of 86^o.

18 A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. The ratio of the number of cubes with exactly two green faces to the number of cubes with three green faces is

(A) 9 : 2 (B) 9 : 4 (C) 6 : 1 (D) 3 : 1 (E) 5 : 2

Solution Option: A The number of cubes with two green faces are cubes which were on the edges between two faces, but not on the corners of the cube, as the corners of the cube will have three green faces. The number of cubes with two green faces are $(2 + 3 + 4) \times 4 = 9 \times 4 = 36$ and the number of corners (and three green faces) are 8. Hence $36 : 8 = 9 : 2$.

19 The sum of the digits of the integer equal to

$$777\ 777\ 777\ 777\ 777^2 - 222\ 222\ 222\ 222\ 223^2$$

is

(A) 148 (B) 84 (C) 74 (D) 69 (E) 79

Solution Option: C Using a difference of squares, the answer is $555\ 555\ 555\ 555\ 554 \times 1\ 000\ 000\ 000\ 000\ 000$ and hence the sum of the digits is $15 \times 5 - 1 = 74$.

20 In trapezoid $ABCD$, AD is parallel to BC. Also, BD is perpendicular to DC. The point F is chosen on line BD so that AF is perpendicular to BD. AF is extended to meet BC at point E. If $AB = 41$, $AD = 50$ and $BF = 9$, what is the area of quadrilateral $FECD$?

(A) 900 (B) 960 (C) 1300 (D) 1523.5 (E) 1560

Solution Option: B By Pythagoras, $AF = \sqrt{41^2 - 9^2} = \sqrt{1681 - 81} = \sqrt{1600} = 40$.
Hence $FD = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$.
Triangle BFE is similar to triangle DFC and hence $FE = 12$.
Triangle BDC is similar to triangle BFE and hence $DC = 52$.
Area of $FECD = \frac{1}{2} \times 52 \times 39 - \frac{1}{2} \times 12 \times 9 = 26 \times 39 - 6 \times 9 = 1014 - 54 = 960$.